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DEMAZURE OPERATORS FOR COMPLEX REFLECTION GROUPS $G(e,e,n)$

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Abstract This paper is a continuation of the work in [RS], where we studied Demazure operators for the imprimitive complex reflection group $\widetilde{W} = G(e,1,n)$ and constructed a homogeneous basis of the coinvariant algebra $S_{\widetilde{W}}$. In this paper, we study a similar problem for the reflection subgroup $W = G(e,e,n)$ of $\widetilde{W}$. We prove, by assuming certain conjectures, that the operators Δ_w $(w \in W)$ are linearly independent over the symmetric algebra $S(V)$. We define a graded space H_W in terms of Demazure operators, and we show that the coinvariant algebra S_W is naturally isomorphic to H_W. Then we can define a homogeneous basis of S_W parametrized by $w \in W$.

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§1. Introduction

Let $\widetilde{W} = G(e,1,n)$ be the imprimitive complex reflection group isomorphic to $S_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$, regarded as a subgroup of $GL(V)$ with $V \cong \mathbb{C}^n$. (Here S_n denotes the symmetric group of degree n). Let $S_{\widetilde{W}}$ be the coinvariant algebra of $\widetilde{W}$, i.e. the quotient of the symmetric algebra $S(V)$ by the ideal generated by the non-constant homogeneous $\widetilde{W}$-invariant polynomials. In [BM1], K. Bremke and G. Malle constructed a length function $n : \widetilde{W} \to \mathbb{N}$ satisfying the property $\sum_{w \in \widetilde{W}} t^{n(w)} = P_{\widetilde{W}}(t)$, where $P_{\widetilde{W}}(t)$ is the Poincaré polynomial associated with the graded algebra $S_{\widetilde{W}}$. In [RS], we defined a Demazure operator Δ_w for each $w \in \widetilde{W}$, which is an endomorphism on $S(V)$ reducing the

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grading by $n(w)$, and constructed a basis of $S_{\widetilde{W}}$ parametrized by $w \in \widetilde{W}$ by making use of $\{\Delta_w | \; w \in \widetilde{W}\}$.

In this paper, we consider the group $W = G(e,e,n)$, which is a subgroup of $\widetilde{W}$ of index e, isomorphic to $S_n \ltimes (\mathbb{Z}/e\mathbb{Z})^{n-1}$. The length function $\ell : W \to \mathbb{N}$, satisfying the property $\sum_{w\in W} t^{\ell(w)} = P_W(t)$, was constructed by [BM2], where $P_W(t)$ is the Poincaré polynomial associated with the coinvariant algebra S_W of W. We recall the definition of Demazure operators. For each $\alpha \in V$, let s_α be the complex reflection with eigenvector α. A Demazure operator $\Delta_\alpha : S(V) \to S(V)$ is defined by

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}, \quad \text{for } f \in S(V).$$

We define an operator Δ_w for each $w \in W$ as follows. It is known by [BM2] that there exists a system of representatives $\mathcal{N}$ of the left cosets W/S_n satisfying the property that $\ell(w'w'') = \ell(w') + \ell(w'')$ for $w' \in \mathcal{N}$, $w'' \in S_n$. We define $\Delta_{w'}$ for $w' \in \mathcal{N}$ as a certain product of various Δ_α for $s_\alpha \in W$. On the other hand, the operator $\Delta_{w''}$ for $w'' \in S_n$ is already defined by the theory of Demazure operators for finite Coxeter groups. Then we define, for $w = w'w'' \in W$ ($w' \in \mathcal{N}$, $w'' \in S_n$) the operator Δ_w by $\Delta_w = \Delta_{w'}\Delta_{w''}$. In the case of $\widetilde{W}$, the crucial step for the proof of the main result is to show that the operators $\{\Delta_w | \; w \in \widetilde{W}\}$ are linearly independent over $S(V)$. In our situation, we can prove (Theorem 3.10) that the operators $\{\Delta_{w'} | \; w' \in \mathcal{N}\}$ are linearly independent over $S(V)$. It is also known by the general theory that the operators $\{\Delta_{w''} | \; w'' \in S_n\}$ are linearly independent over $S(V)$. We expect that $\{\Delta_w | \; w \in W\}$ are linearly independent over $S(V)$. In our paper, we prove this by assuming certain conjectures, (3.12.1) and (3.12.2), concerning the property of $\Delta_{w'}$ ($w' \in \mathcal{N}$). Our main result asserts that a similar theorem as in the case of $\widetilde{W}$ holds for W, assuming the above conjectures. More precisely, let $\bar{\mathcal{D}}_W$ be the subspace of the dual space of $S(V)$ generated by $\varepsilon\Delta_w$ ($w \in W$), where $\varepsilon : S(V) \to \mathbb{C}$ is the evaluation at 0. Then we can show (Theorem 3.25) that $\{\varepsilon\Delta_w | \; w \in W\}$ gives a basis of $\bar{\mathcal{D}}_W$, and that S_W is naturally isomorphic to the dual space of $\bar{\mathcal{D}}_W$.

The conjecture (3.12.1) is related to the evaluation of Δ_{w_1} (w_1 is the longest element in W with respect to ℓ) at certain polynomial, and is verified to be true (Theorem 3.14) under the assumption that $e \geq n$. This theorem leads to the following interesting characterization of Δ_{w_1}. Let J be the operator on $S(V)$ defined by $J = \sum_{w\in W} \varepsilon_W(w)w$, where $\varepsilon_W : W \to \{\pm 1\}$ is the sign character of W. Let Q be the product of all eigenvectors of reflections contained in W. Assume that $e \geq n$. Then Δ_{w_1} is expressed (Proposition 3.18) as $\Delta_{w_1} = dQ^{-1}J$ for some non-zero constant $d \in \mathbb{C}$.

§2. Preliminaries

2.1. Let V be the unitary space $\mathbb{C}^n$ with standard basis $x_1, x_2, \dots, x_n$. Let $\widetilde{W} = G(e,1,n)$ be the imprimitive complex reflection group contained in $GL(V)$. The group $\widetilde{W}$ is generated by $\{t, s_2, \cdots, s_n\}$, where s_i is a reflection permuting x_i and x_{i-1}, and t is a complex reflection of order e, which sends x_1 to ζx_1 and leaves all the other x_i unchanged. (Here ζ is a fixed primitive e-th root of unity).

Let $W = G(e,e,n)$ be the subgroup of $\widetilde{W}$ of index e generated by reflections $S = \{s_1, s_2, \cdots, s_n\}$ of order 2, where $s_1 = ts_2t^{-1}$ sends x_1 to $\zeta^{-1}x_2$ and x_2 to ζx_1. Note that W is the Weyl group of type D_n if $e = 2$, and W is the dihedral group of order $2e$ if $n = 2$.

Let $S(V) = \oplus_{i\geq 0} S^i(V)$ be the symmetric algebra on V, where $S^i(V)$ denotes the i-th homogeneous part of $S(V)$. The group W acts naturally on $S(V)$ and we denote by I_W the ideal of $S(V)$ generated by the W-invariant homogeneous elements of $S(V)$ of strictly positive degree. The coinvariant algebra associated with W is defined as $S_W = S(V)/I_W$, which has a natural grading $S_W = \oplus_{i\geq 0} S^i_W$ inherited from that of $S(V)$. The Poincaré polynomial $P_W(t)$ is defined by the formula

$$P_W(t) = \sum_{i\geq 0} \dim_{\mathbb{C}}(S^i_W)t^i.$$

The group $\widetilde{W}$ acts on $S(V)$, and the coinvariant algebra $S_{\widetilde{W}}$ and the Poincaré polynomial $P_{\widetilde{W}}(t)$ associated with $\widetilde{W}$ are defined similarly.

2.2. In [BM1], Bremke and Malle constructed a length function $n : \widetilde{W} \to \mathbb{N}$ by making use of a certain root system, and showed that the sum $\sum_{w\in\widetilde{W}} t^{n(w)}$ coincides with $P_{\widetilde{W}}(t)$. In [BM2], they defined a different type of length function $\ell : \widetilde{W} \to \mathbb{N}$, (the function ℓ_2 in the notation of [BM2]), in terms of an alternative root system and showed that the restriction of ℓ on W satisfies the formula $\sum_{w\in W} t^{\ell(w)} = P_W(t)$. Note that the subgroup of W generated by $S' = \{s_2, \cdots, s_n\}$ is identified with S_n. The restriction of ℓ on S_n coincides with the usual length function of S_n with respect to S'.

They found a system of left coset representatives $\mathcal{N}$ of W/S_n having nice properties with respect to the length function ℓ on W as follows. For $0 < a \leq e$, $1 \leq i \leq n$ we define an element of $\widetilde{W}$ by

$$w(a,i) = \begin{cases} s_i \cdots s_2 t^a & \text{if } 0 < a \leq e/2, \\ s_i \cdots s_2 t^a s_2 \cdots s_i & \text{if } e/2 < a \leq e. \end{cases} \tag{2.2.1}$$

It is known by Lemma 1.10 in [BM2] that the length of the element $w(a,i)$ is given as

$$\ell(w(a,i)) = \begin{cases} (i-1)(2a-1) & \text{if } 0 < a \le e/2, \\ (i-1)(2e-2a) & \text{if } e/2 < a \le e. \end{cases} \tag{2.2.2}$$

Put

$$\mathcal{N} = \{w(a_1,1)\cdots w(a_n,n) |\ 1 \le a_i \le e,\ \sum_{i=1}^{n} a_i \equiv 0 \pmod e\}$$

They proved the following fact.

Proposition 2.3 ([BM2, Cor.1.16, Prop. 2.6]). *The set $\mathcal{N}$ is a system of representatives for the left cosets W/S_n satisfying the following.*

(i) For $w' \in \mathcal{N}$, $w'' \in S_n$, we have

$$\ell(w'w'') = \ell(w') + \ell(w'').$$

(ii) If $w' \in \mathcal{N}$ is given as $w' = w(a_1,1)\cdots w(a_n,n)$, then $\ell(w') = \sum_{i=2}^{n} \ell(w(a_i,i))$.

(Note that $\ell(w(a_1,1)) = 0$ by (2.2.2)).

2.4. Let s_α be the reflection in W with eigenvector $\alpha \in V$. (Here we assume that the eigenvalue attached to α is not equal to 1). We define an operator $\Delta_\alpha : S(V) \to S(V)$ by the formula

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}, \quad (f \in S(V)).$$

We call Δ_α a Demazure operator on $S(V)$. Demazure operators are defined for complex reflection groups in general. In the case of finite Coxeter groups, there exists a well established theory for Demazure operators by [BBG], [D]. In the case of (non-real) finite complex reflection groups, not much is known. In [RS], we studied Demazure operators for the group $\widetilde{W}$, and showed that the structure of the coinvariant algebra $S_{\widetilde{W}}$ is described in terms of Demazure operators, as in the case of Coxeter groups, by constructing a certain (non-canonical) basis of $S_{\widetilde{W}}$. Here we take up a similar problem for the group W.

We give some properties of Demazure operators. We have the following.

$$\Delta_\alpha^2 = 0, \tag{2.4.1}$$

$$\Delta_\alpha(fh) = \Delta_\alpha(f)h + f\Delta_\alpha(h), \tag{2.4.2}$$

for $f, h \in S(V)$. If $f \in S(V)$ is s_α-invariant, then $\Delta_\alpha(f) = 0$. Now let $S(V)^W$ be the subalgebra of $S(V)$ consisting of the W-invariant elements. Then it follows from (2.4.2) that

$$\Delta_\alpha(fh) = f\Delta_\alpha(h) \qquad \text{for } f \in S(V)^W. \tag{2.4.3}$$

In particular, we have $\Delta_\alpha(I_W) \subseteq I_W$ and Δ_α induces an operation on S_W.

2.5. Let S_n be the subgroup of W as in 2.2. Then (S_n, S') is a Coxeter system, with associated length function $\ell : S_n \to \mathbb{N}$. Hence, by the general theory of Demazure operators for finite Coxeter groups, we have the following facts. Let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ $(s_i \in S')$ be a reduced expression of $w \in S_n$. Then we define

$$\Delta_w = \Delta_{i_1}\cdots\Delta_{i_k}, \tag{2.5.1}$$

where $\Delta_i = \Delta_{\alpha_i}$ with $\alpha_i = x_i - x_{i-1}$. It is known that the operator Δ_w is independent of the choice of the reduced expression. (See, for example [H, IV, Prop. 1.7]).

Let w_0 be the longest element in S_n. We define a polynomial Q_0 by $Q_0 = \prod_{i>j}(x_i - x_j)$. The following facts are known.

Proposition 2.6 ([H, IV, Prop. 1.6]). $\Delta_{w_0}(Q_0) = 1$.

Proposition 2.7 ([H, IV, Cor. 2.3]). *For any $w, w' \in W$ such that $\ell(w) \le \ell(w')$, we have $\Delta_{w'}\Delta_{w^{-1}w_0} = \delta_{w,w'}\Delta_{w_0}$.*

Note that the condition $\ell(w) \le \ell(w')$ is dropped in the statement of Corollary 2.3 in [H].

§3. Demazure operators for $G(e, e, n)$

3.1. From now on we identify $S(V)$ with the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ with indeterminates x_i. The group $W = G(e, e, n)$ acts on $\mathbb{C}[x_1, \dots, x_n]$ as in 2.1.

For $i = 2, 3, \dots, n$ we define inductively the element s'_i as follows; Let $s'_2 = s_1$ and $s'_i = s_{i-1}s_is'_{i-1}s_is_{i-1}$. Then s'_i is the complex reflection of order 2, which sends x_i to ζx_{i-1}, and x_{i-1} to $\zeta^{-1}x_i$. We note that if we put $y_i = \zeta^{-1/2}x_i$ and $y_{i-1} = \zeta^{1/2}x_{i-1}$, then we can regard s'_i as a permutation of y_i, y_{i-1}. We define two operators Δ_{s_i}, $\Delta_{s'_i}$ on $S(V)$ by the formulas

(3.1.1)

$$\Delta_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i-1}}, \qquad \Delta_{s'_i}(f) = \frac{f - s'_i(f)}{\zeta^{-1/2}x_i - \zeta^{1/2}x_{i-1}}, \qquad (f \in S(V)).$$

Then the following two formulas hold:

$$(3.1.2)\qquad \begin{aligned}\Delta_{s_i}(x_i^a x_{i-1}^b) &= \varepsilon \sum x_i^j x_{i-1}^{a+b-1-j},\\ \Delta_{s_i'}(x_i^a x_{i-1}^b) &= \varepsilon \zeta^{(2a-1)/2} \sum \zeta^{-j} x_i^j x_{i-1}^{a+b-1-j},\end{aligned}$$

where in both formulas the sum is taken over j such that $\min\{a,b\} \leq j \leq \max\{a,b\}-1$, and $\varepsilon = 1$ (resp. $\varepsilon = -1$) if $a > b$, (resp. $a < b$). The first formula is contained in [RS], and the second one is obtained from the first by changing the variables $x_i \mapsto y_i$, $x_{i-1} \mapsto y_{i-1}$.

For $i = 2, \cdots, n$, we define operators $\Delta_i^{(a)}$, $\Delta_{i'}^{(a)}$ in the following way

$$(3.1.3)\qquad \Delta_i^{(a)} = \underbrace{\cdots \Delta_{s_i'} \Delta_{s_i}}_{a-\text{factors}}, \qquad \Delta_{i'}^{(a)} = \underbrace{\cdots \Delta_{s_i} \Delta_{s_i'}}_{a-\text{factors}}.$$

3.2. In order to study the above operators in a more detailed way, we need to evaluate them at various polynomials. For this we prepare some notation. Let a, b be two positive integers such that $1 \leq a \leq b$. We put

$$c(a,b) = (-1)^{[a+1/2]} \prod_{j=1}^{a-1} (\zeta^{(b-j)/2} - \zeta^{-(b-j)/2}),$$

where $[a]$ denotes the smallest integer which does not exceed a. We have $c(a,b) = -1$ if $a = 1$. The following two lemmas will be used in our later discussion.

Lemma 3.3. *Let a,b be integers such that $1 \leq a \leq b$.*

(i) Assume that $a < b$. Then we have

$$\Delta_i^{(a)}(x_{i-1}^b) = \begin{cases} c(a,b)(x_i^{b-a} + x_{i-1}^{b-a}) + f, & \text{if } a \text{ is odd},\\ c(a,b)(y_i^{b-a} + y_{i-1}^{b-a}) + f, & \text{if } a \text{ is even},\end{cases}$$

$$\Delta_{i'}^{(a)}(x_{i-1}^b) = \begin{cases} (-1)^{a-1}\zeta^{-b/2} c(a,b)(y_i^{b-a} + y_{i-1}^{b-a}) + f, & \text{if } a \text{ is odd},\\ (-1)^{a-1}\zeta^{-b/2} c(a,b)(x_i^{b-a} + x_{i-1}^{b-a}) + f, & \text{if } a \text{ is even},\end{cases}$$

where in each case, f denotes a polynomial divisible by $x_i x_{i-1} = y_i y_{i-1}$.

(ii) Assume that $a = b$. Then we have

$$\begin{aligned}\Delta_i^{(a)}(x_{i-1}^a) &= c(a,a),\\ \Delta_{i'}^{(a)}(x_{i-1}^a) &= (-1)^{a-1}\zeta^{-a/2} c(a,a).\end{aligned}$$

Proof. We prove only the formula (i). The proof of (ii) is similar, and simpler. We show the first formula in (i). The case where $a = 1$ is straightforward from (3.1.2). The following two formulas are obtained by using the definition of Δ_{s_i}, $\Delta_{s'_i}$ and the fact that $y_i = \zeta^{-1/2}x_i$ and $y_{i-1} = \zeta^{1/2}x_{i-1}$.

$$\begin{aligned}
\Delta_{s'_i}(x_i^{b-a+1} + x_{i-1}^{b-a+1}) &= (\zeta^{(b-a+1)/2} - \zeta^{-(b-a+1)/2})(y_i^{b-a} + y_{i-1}^{b-a}) + f_1, \\
\Delta_{s_i}(y_i^{b-a+1} + y_{i-1}^{b-a+1}) &= (\zeta^{-(b-a+1)/2} - \zeta^{(b-a+1)/2})(x_i^{b-a} + x_{i-1}^{b-a}) + f_1,
\end{aligned}$$

where f_1 is a polynomial divisible by $x_ix_{i-1} = y_iy_{i-1}$. We also notice that since $x_ix_{i-1} = y_iy_{i-1}$ is stable by the reflections s_i and s'_i, if a polynomial f is divisible by $x_ix_{i-1} = y_iy_{i-1}$, then so are $\Delta_{s_i}(f)$ and $\Delta_{s'_i}(f)$. The first formula in (i) follows from the above formulas by induction on a. Next we show the second formula in (i). If we note that $x_{i-1}^b = \zeta^{-b/2}y_{i-1}^b$, it is easy to see that $\Delta_{i'}^{(a)}(y_{i-1}^b)$ coincides with the polynomial which is obtained from $\Delta_i^{(a)}(x_{i-1}^b)$ by replacing x_i, x_{i-1} by y_i, y_{i-1}, by replacing ζ by ζ^{-1}, and then by multiplying by $\zeta^{-b/2}$. Hence the second formula follows immediately from the first one. $\square$

Next we compute the values $\Delta_i^{(a)}(x_i^b)$ and $\Delta_{i'}^{(a)}(x_i^b)$. By (3.1.2) we see that

$$\Delta_{s_i}(x_i^b) = -\Delta_{s_i}(x_{i-1}^b), \qquad \Delta_{s'_i}(y_i^b) = -\Delta_{s'_i}(y_{i-1}^b).$$

Therefore we have

$$\begin{aligned}
\Delta_{s'_i}(x_i^b) &= \zeta^{b/2}\Delta_{s'_i}(y_i^b) \\
&= -\zeta^{b/2}\Delta_{s'_i}(y_{i-1}^b) \\
&= -\zeta^{b}\Delta_{s'_i}(x_{i-1}^b).
\end{aligned}$$

This implies that the value $\Delta_i^{(a)}(x_i^b)$ (resp. $\Delta_{i'}^{(a)}(x_i^b)$) coincides with $-\Delta_i^{(a)}(x_{i-1}^b)$ (resp. $-\zeta^b\Delta_{i'}^{(a)}(x_{i-1}^b)$). Therefore as a corollary to Lemma 3.3 we obtain the following result.

Lemma 3.4. *Let a, b as in Lemma 3.3.*

(i) Assume that $a < b$. Then we have

$$\Delta_i^{(a)}(x_i^b) = \begin{cases} -c(a,b)(x_i^{b-a} + x_{i-1}^{b-a}) + f & \text{if } a \text{ is odd}, \\ -c(a,b)(y_i^{b-a} + y_{i-1}^{b-a}) + f & \text{if } a \text{ is even}, \end{cases}$$

$$\Delta_{i'}^{(a)}(x_i^b) = \begin{cases} (-1)^a\zeta^{b/2}c(a,b)(y_i^{b-a} + y_{i-1}^{b-a}) + f & \text{if } a \text{ is odd}, \\ (-1)^a\zeta^{b/2}c(a,b)(x_i^{b-a} + x_{i-1}^{b-a}) + f & \text{if } a \text{ is even}. \end{cases}$$

(ii) Assume that $a = b$. Then we have

$$\Delta_i^{(a)}(x_i^a) = -c(a,a),$$
$$\Delta_{i'}^{(a)}(x_i^a) = (-1)^a\zeta^{a/2}c(a,a).$$

3.5. We fix an integer $a \geq 0$. We define, for $2 \leq i \leq n$, an operator $\Delta_i[a]$ on $S(V)$ by the formula

$$\Delta_i[a] = \begin{cases} \Delta_{2'}^{(a)} \cdots \Delta_{i'}^{(a)} & \text{if } a \geq 1, \\ 1 & \text{if } a = 0. \end{cases}$$

The operator $\Delta_i[a]$ reduces the grading by $(i-1)a$. For each $a \geq 0$, we define a polynomial $g_{i,a}(x)$ of degree $(i-1)a$ by $g_{i,a}(x) = (x_1 \cdots x_{i-1})^a$. Then the following lemma holds.

Lemma 3.6. *Assume that $a \geq 1$. Let $\Delta_i[a]$, $g_{i,a}(x)$ be defined as above. Then*

$$\Delta_i[a](g_{i,a}) = \{(-1)^{a-1}\zeta^{-a/2}c(a,a)\}^{i-1}.$$

In particular, $\Delta_i[a](g_{i,a}) \neq 0$ for $1 \leq a \leq e-1$.

Proof. First we note that the operator $\Delta_{i'}^{(a)}$ affects only the variables x_i and x_{i-1} and leaves all the others unchanged. Therefore we have

$$\Delta_i[a](g_{i,a}) = (x_1 \cdots x_{i-2})^a \Delta_{i'}^{(a)}(x_{i-1}^a). \tag{3.6.1}$$

But we have $\Delta_{i'}^{(a)}(x_{i-1}^a) = (-1)^{a-1}\zeta^{-a/2}c(a,a)$ by Lemma 3.3 (ii). Hence the right hand side of (3.6.1) can be written as $\gamma g_{i-1,a}$ with $\gamma = (-1)^{a-1}\zeta^{-a/2}c(a,a)$. Repeating this procedure for the operators $\Delta_{(i-1)'}^{(a)}, \cdots, \Delta_{2'}^{(a)}$ we obtain the result. □

3.7. Let $\mathcal{M} = [0, e-1]^{n-1}$ ($n-1$ copies of the interval $[0, e-1]$). For each $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathcal{M}$, we define an operator Δ_λ on $S(V)$ by

$$\Delta_\lambda = \Delta_n[\lambda_n] \cdots \Delta_2[\lambda_2].$$

Also for $\lambda \in \mathcal{M}$ we define a polynomial $P_\lambda(x)$ by $P_\lambda = \prod_{i=2}^n g_{i,\lambda_i}$. Let $\lambda = (\lambda_2, \cdots, \lambda_n)$, $\mu = (\mu_2, \cdots, \mu_n) \in \mathcal{M}$. We define a total order $\lambda > \mu$ on $\mathcal{M}$ by $\lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1}$ and $\lambda_i > \mu_i$ for some $i \geq 1$. Then we have the following proposition.

Proposition 3.8. *Let $\lambda, \mu \in \mathcal{M}$. Then there exists a non-zero element $c_\lambda \in \mathbb{C}$ such that*

$$\Delta_\lambda(P_\mu) = \begin{cases} c_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda > \mu. \end{cases}$$

Proof. First we note that $\Delta_j[\lambda_j]$ leaves $g_{i,\mu_i} = (x_1 \cdots x_{i-1})^{\mu_i}$ invariant for $j < i$. In fact, $\Delta_j[\lambda_j]$ consists of various products of the operators $\Delta_{s_2}, \cdots, \Delta_{s_j}, \Delta_{s'_2} \cdots, \Delta_{s'_j}$ and these operators leave g_{i,μ_i} invariant, since s_j and s'_j stabilize $x_{j-1}x_j = y_{j-1}y_j$ (in the notation of 3.1).

First assume that $\lambda = \mu$. Then by Lemma 3.6 $\Delta_i[\lambda_i](g_{i,\lambda_i})$ is a non-zero constant for each i. Combining with the above remark, we see that

$$\Delta_\lambda(P_\lambda) = \prod_{i=2}^{n} \Delta_i[\lambda_i](g_{i,\lambda_i}),$$

and the right hand side is a non-zero constant, which we write as c_λ.
Next assume that $\lambda > \mu$. Then there exists i such that $\lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1}$ and $\lambda_i > \mu_i$. Then we have

$$\Delta_\lambda(P_\mu) = c\Delta_n[\lambda_n] \cdots \Delta_i[\lambda_i](\prod_{j=i}^{n} g_{j,\mu_j}),$$

with some $c \in \mathbb{C} - \{0\}$ by a similar argument as in the previous case. But then

$$\Delta_i[\lambda_i](\prod_{j=i}^{n} g_{j,\mu_j}) = (\prod_{j=i+1}^{n} g_{j,\mu_j})\Delta_i[\lambda_i](g_{i,\mu_i}),$$

and $\Delta_i[\lambda_i](g_{i,\mu_i}) = 0$, since $\Delta_i[\lambda_i]$ reduces the degree by $(i-1)\lambda_i$, which is bigger than the degree of g_{i,μ_i}. Hence $\Delta_\lambda(P_\mu) = 0$. □

3.9. Let $\mathcal{D}_W$ be the subalgebra of $\operatorname{End}_{\mathbb{C}} S(V)$ generated by Δ_s ($s \in S$) and α^* ($\alpha \in V$), where $\alpha^* : S(V) \to S(V)$ denotes the multiplication by the vector α. Then $\mathcal{D}_W$ becomes a left $S(V)$-module. We also note that for any $w \in W$ the endomorphism w on $S(V)$ is contained in $\mathcal{D}_W$, since $s_\alpha = 1 - \alpha^*\Delta_\alpha \in \mathcal{D}_W$ for any $s_\alpha \in S$. Since $\Delta_{s'_i} = w\Delta_{s'_2}w^{-1}$ for some $w \in S_n$, we see that $\Delta_{s'_i}$ ($2 \le i \le n$) are also contained in $\mathcal{D}_W$. Therefore $\Delta_\lambda \in \mathcal{D}_W$ for any $\lambda \in \mathcal{M}$. As a corollary to Proposition 3.8 we have the following theorem. The proof is immediate from Proposition 3.8.

Theorem 3.10. *The set $\{\Delta_\lambda |\ \lambda \in \mathcal{M}\}$ of operators in $\mathcal{D}_W$ is linearly independent over $S(V)$.*

3.11. In the case of $\widetilde{W} = G(e, 1, n)$, the operator Δ_w was constructed in [RS] for each $w \in \widetilde{W}$ by making use of a particular reduced expression of w. Here Δ_w is an operator which reduces the grading by $n(w)$. In our case, the operators Δ_λ with $\lambda \in \mathcal{M}$ are not directly related to the elements of W. However, one gets a bijection between the set $\{\Delta_\lambda | \lambda \in \mathcal{M}\}$ and the set $\mathcal{N}$ in W as follows. For each $0 < a \le e$, we set

$$\varphi(a) = \begin{cases} 2a - 1 & \text{if } 0 < a \le e/2, \\ 2e - 2a & \text{if } e/2 < a \le e. \end{cases}$$

Then the map φ gives rise to a bijection from the set $[1,\, e]$ to the set $[0,\, e-1]$, and one can define a bijection $\widetilde{\varphi} : \mathcal{N} \to \mathcal{M}$ by $\widetilde{\varphi}(w) = (\varphi(a_2), \dots, \varphi(a_n))$. Hence the set $\{\Delta_\lambda | \lambda \in \mathcal{M}\}$ is in bijection with the set $\mathcal{N}$. It is easily checked, by using (2.2.2), that if $\lambda \in \mathcal{M}$ corresponds to $w \in \mathcal{N}$, then Δ_λ reduces the degree by $\ell(w)$.

3.12. In the case of $\widetilde{W}$, it was shown in [RS, Prop. 2.14] that $\mathcal{D}_{\widetilde{W}}$ is a free S(V)-module with basis $\{\Delta_w |\ w \in \widetilde{W}\}$. In order to obtain a similar result for W, we try to construct operators Δ_w for any $w \in W$. In view of Proposition 2.3, any element $w \in W$ can be expressed uniquely as $w = w'w''$, with $w' \in \mathcal{N}$, $w'' \in S_n$ with $\ell(w) = \ell(w') + \ell(w'')$. We now define Δ_w $(w \in W)$ by $\Delta_w = \Delta_\lambda \Delta_{w''}$, where $\lambda \in \mathcal{M}$ is given by $\lambda = \widetilde{\varphi}(w')$. (Note that the operator $\Delta_{w''}$ corresponding to $w'' \in S_n$ is defined without ambiguity, see 2.5).

We know, by Theorem 3.10, that the set $\{\Delta_\lambda |\ \lambda \in \mathcal{M}\}$ is linearly independent over $S(V)$. It is also known that the set $\{\Delta_{w''} |\ w'' \in S_n\}$ is linearly independent over $S(V)$. We expect that the set $\{\Delta_w |\ w \in W\}$ gives rise to a basis of $\mathcal{D}_W$. In what follows, we show that this conjecture is reduced to some properties of Δ_λ. Here we prepare some notation. For each $\lambda \in \mathcal{M}$ we define the length $\ell(\lambda)$ by $\ell(\lambda) = \ell(w')$ whenever λ corresponds to $w' \in \mathcal{N}$. Hence $\ell(w) = \ell(\lambda) + \ell(w'')$ if $w \in W$ corresponds to the pair $(\lambda, w'') \in \mathcal{M} \times S_n$. For each integer $c \ge 1$, we put $\mathcal{M}_c = \{\lambda \in \mathcal{M} |\ \ell(\lambda) = c\}$. For each polynomial P_λ $(\lambda \in \mathcal{M})$ given in 3.7, we define its average $\widetilde{P}_\lambda$ over S_n by $\widetilde{P}_\lambda = \sum_{\sigma \in S_n} \sigma(P_\lambda)$. Note that $\Delta_\lambda(\widetilde{P}_\mu)$ is a constant if $\lambda, \mu \in \mathcal{M}_c$ for some c. Let $\lambda_0 = (e-1, \cdots, e-1) \in \mathcal{M}$. Then λ_0 is the longest element in $\mathcal{M}$ with $\ell(\lambda_0) = n(n-1)(e-1)/2$. We consider the following two statements.

(3.12.1) $\Delta_{\lambda_0}(\widetilde{P}_{\lambda_0})$ is a non-zero constant.

(3.12.2) For any integer $c \ge 1$, the matrix $(\Delta_\lambda(\widetilde{P}_\mu))_{\lambda,\mu \in \mathcal{M}_c}$ is non-singular.

We don't know whether these two statements hold in a full generality for W. It is verified that (3.12.1) holds whenever $e \ge n$, which will be discussed in Theorem 3.14. In the case where $n = 3$ it is checked that (3.12.2) holds for small e. Note that (3.12.1) is a special case of (3.12.2), since the set $\mathcal{M}_c$ consists of a single element λ_0 if $c = \ell(\lambda_0)$.

3.13. In order to look at $\widetilde{P}_\lambda$ more precisely, we shall extend the parameter set $\mathcal{M}$ to $\mathbb{N}^{n-1}$. For each $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathbb{N}^{n-1}$, we define a polynomial $F_n(\lambda)$ by $F_n(\lambda) = \prod_{i=2}^n g_{i,\lambda_i}$. Hence if $\lambda \in \mathcal{M}$, $F_n(\lambda)$ coincides with P_λ. We put $\widetilde{F}_n(\lambda) = \sum_{\sigma\in S_n} \sigma(F_n(\lambda))$.
For each i $(1 \le i \le n)$, let

$$\sigma_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & n & i & i+1 & \cdots & n-1 \end{pmatrix} \in S_n.$$

Then $\{\sigma_1, \cdots, \sigma_n\}$ is a complete set of representatives of the right cosets $S_{n-1}\backslash S_n$. For each $\mu = (\mu_2, \cdots, \mu_n) \in \mathbb{N}^{n-1}$, we define $\mu^{(i)} \in \mathbb{N}^{n-2}$, $(2 \le i \le n-1)$ by

$$\mu^{(i)} = (\mu_2, \cdots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \cdots, \mu_n).$$

Also we put $\mu^{(1)} = (\mu_3, \cdots, \mu_n) \in \mathbb{N}^{n-2}$ and $\mu^{(n)} = (\mu_2, \cdots, \mu_{n-1}) \in \mathbb{N}^{n-2}$. Then it is easy to see that

$$\sigma_i(F_n(\mu)) = \begin{cases} F_{n-1}(\mu^{(i)}) \cdot x_n^{b_i(\mu)} & \text{if } 1 \le i \le n-1, \\ F_{n-1}(\mu^{(n)}) \cdot (x_1 \cdots x_{n-1})^{\mu_n} & \text{if } i = n, \end{cases} \tag{3.13.1}$$

where $b_i(\mu) = \mu_{i+1} + \cdots + \mu_n$ for $i = 1, \cdots, n-1$. It follows from (3.13.1) that

$$\sum_{\sigma\in S_{n-1}} \sigma\sigma_i F_n(\mu) = \begin{cases} \widetilde{F}_{n-1}(\mu^{(i)}) \cdot x_n^{b_i(\mu)} & \text{if } 1 \le i \le n-1, \\ \widetilde{F}_{n-1}(\mu^{(n)}) \cdot (x_1 \cdots x_{n-1})^{\mu_n} & \text{if } i = n. \end{cases}$$

Hence we have a recursive formula,

$$\widetilde{F}_n(\mu) = \sum_{i=1}^{n-1} \widetilde{F}_{n-1}(\mu^{(i)}) x_n^{b_i(\mu)} + \widetilde{F}_{n-1}(\mu^{(n)})(x_1 \cdots x_{n-1})^{\mu_n}. \tag{3.13.2}$$

Let $\mathcal{M}' = [0, e-1]^{n-2}$ be the set corresponding to the situation in $G(e, e, n-1)$. Then for $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathcal{M}$, the operator Δ_λ can be written as $\Delta_\lambda = \Delta_n[\lambda_n]\Delta_{\lambda'}$ with $\lambda' = (\lambda_2, \cdots, \lambda_{n-1}) \in \mathcal{M}'$. By applying Δ_λ to the formula (3.13.2), we obtain

$$\begin{aligned} \Delta_\lambda(\widetilde{F}_n(\mu)) = \sum_{i=1}^{n-1} & \Delta_n[\lambda_n](\Delta_{\lambda'}(\widetilde{F}_{n-1}(\mu^{(i)})) \cdot x_n^{b_i(\mu)}) \\ & + \Delta_n[\lambda_n](\Delta_{\lambda'}(\widetilde{F}_{n-1}(\mu^{(n)})) \cdot (x_1 \cdots x_{n-1})^{\mu_n}). \end{aligned} \tag{3.13.3}$$

By making use of the formula (3.13.3), we can compute the value $\Delta_{\lambda_0}(\widetilde{P}_{\lambda_0})$ under a certain condition, which gives a partial answer to the conjecture (3.12.1).

Theorem 3.14. *Assume that $e \geq n$. Then $\Delta_{\lambda_0}(\widetilde{P}_{\lambda_0}) = c_{\lambda_0}$, where c_{λ_0} is given as in Proposition 3.8.*

Proof. Since $\lambda_0 = (e-1, \cdots, e-1) \in \mathcal{M}$, Δ_{λ_0} can be written as $\Delta_{\lambda_0} = \Delta_{n-1}[e-1]\Delta_{\lambda_0'}$, where $\lambda_0' = (e-1, \cdots, e-1) \in \mathcal{M}'$. First we note the following

(3.14.1) Let $\mu = (\mu_2, \cdots, \mu_n) \in \mathbb{N}^{n-1}$. Assume that $\mu_i \equiv 0 \pmod{e-1}$ for all i, and that $e-1 < \sum_i \mu_i < e(e-1)$. Then we have $\Delta_{\lambda_0}(\widetilde{F}_n(\mu)) = 0$.

We prove (3.14.1) by induction on n. We apply the formula (3.13.3) with $\lambda = \lambda_0$. Note that if μ satisfies the assumption of (3.14.1), then $\mu^{(i)}$ $(2 \leq i \leq n-1)$ above also satisfies the same condition. Hence (3.13.3) implies, by induction hypothesis, that

$$\begin{aligned}\Delta_{\lambda_0}(\widetilde{F}_n(\mu)) = \Delta_n[e-1](\Delta_{\lambda_0'}(\widetilde{F}_{n-1}(\mu^{(1)})) \cdot x_n^{b_1(\mu)}) \\ + \Delta_n[e-1](\Delta_{\lambda_0'}(\widetilde{F}_{n-1}(\mu^{(n)})) \cdot (x_1 \cdots x_{n-1})^{\mu}).\end{aligned}$$

Here we may assume that $\mu^{(1)} = \lambda_0'$ or $\mu^{(n)} = \lambda_0'$, since both of $\Delta_{\lambda_0'}(\widetilde{F}_{n-1}(\mu^{(1)}))$ and $\Delta_{\lambda_0'}(\widetilde{F}_{n-1}(\mu^{(n)}))$ are zero, otherwise. But if $\mu^{(1)} = \lambda_0'$, then $\widetilde{F}_1(\mu^{(n)}) = \widetilde{P}_{\lambda_0'}$, and $\Delta_{\lambda_0'}(\widetilde{P}_{\lambda_0'})$ is a constant. The same argument holds for the case $\mu^{(n)} = \lambda_0'$. Therefore, in order to prove (3.14.1), we have only to show that

(3.14.2) $\Delta_n[e-1]x_n^{b_1(\mu)} = 0$,

(3.14.3) $\Delta_n[e-1](x_1 \cdots x_{n-1})^{\mu_n} = 0$.

The left hand side of (3.14.2) can be computed by making use of the formula in Lemma 3.4. In particular, it is divisible by $c(e-1, b_1(\mu))$. We claim that $c(e-1, b_1(\mu)) = 0$. In fact, by our assumption, $b_1(\mu) = \mu_2 + \cdots + \mu_n$ can be written as $b_1(\mu) = d(e-1)$ for some d such that $1 < d < e$. Then there exists j $(1 \leq j \leq e-2)$ such that $b_1(\mu) - j \equiv 0 \pmod{e}$. This implies that $c(e-1, b_1(\mu)) = 0$, and (3.14.2) holds. (3.14.3) can be proved in a similar way, by replacing $b_1(\mu)$ by μ_n, and by using Lemma 3.3. Hence (3.14.1) is proved.

We now prove the theorem. We compute $\Delta_{\lambda_0}(\widetilde{P}_{\lambda_0})$ by applying (3.13.3) with $\lambda_0 = \mu$. Then $\lambda_0^{(i)}$ $(2 \leq i \leq n-1)$ satisfies the condition in (3.14.1), since $(n-1)(e-1) < e(e-1)$ by our assumption. Hence, by applying (3.14.1), the terms corresponding to $\mu^{(i)}$ $(2 \leq i \leq n-1)$ vanish. It follows that

$$\begin{aligned}\Delta_{\lambda_0}(\widetilde{P}_{\lambda_0}) = \Delta_n[e-1]x_n^{(n-1)(e-1)} \cdot \Delta_{\lambda_0'}(\widetilde{P}_{\lambda_0'}) \\ + \Delta_n[e-1](x_1 \cdots x_{n-1})^{e-1} \cdot \Delta_{\lambda_0'}(\widetilde{P}_{\lambda_0'}).\end{aligned}$$

But the first term of the sum goes to 0 by applying (3.14.2) with $\mu = \lambda_0$.

Since $(x_1 \cdots x_{n-1})^{e-1} = g_{n,e-1}$, the second term coincides with c_{λ_0}, by Proposition 3.8. This proves the theorem. □

3.15. Let $w_0 \in S_n$ be as in 2.5, and let $w_1 \in W$ be the element in W corresponding to $(\lambda_0, w_0) \in \mathcal{M} \times S_n$. Then w_1 is the longest element in W with $\ell(w_1) = en(n-1)/2 = N$, where N is the number of reflections in W. Let Q_0 be as in 2.5. Then $\widetilde{P}_{\lambda_0} Q_0$ is a polynomial of degree N. Since $\widetilde{P}_\lambda$ is S_n-invariant, and $\Delta_{w_0}(Q_0) = 1$ by Proposition 2.6, we have

$$\Delta_{\lambda_0} \Delta_{w_0}(\widetilde{P}_{\lambda_0} Q_0) = \Delta_{\lambda_0}(\widetilde{P}_{\lambda_0}) = c_{\lambda_0}. \tag{3.15.1}$$

Before stating the next result, we prepare a simple lemma.

Lemma 3.16. *Let $\varepsilon : S(V) \to \mathbb{C}$ denotes the evaluation at 0. Let I_W be the ideal of $S(V)$ defined in 2.3. Then for any $w \in W$ we have*

$$\varepsilon \Delta_w(I_W) = 0$$

Proof. Let f be an element of I_W. Then f can be written as

$$f = \sum_i u_i f_i,$$

with $u_i \in S(V)$, $f_i \in S(V)^W$, where f_i is homogeneous of positive degree. Then applying Δ_w to f, we obtain

$$\Delta_w(f) = \sum \Delta_w(u_i) f_i,$$

since f_i is W-invariant. Here $\Delta_w(u_i) f_i$ is a polynomial without a constant term. This implies that $\varepsilon \Delta_w(f) = 0$ and the lemma follows. □

3.17. Let $\varepsilon_W : W \to \{\pm 1\}$ be the sign character of W. Let Q be the polynomial in $\mathbb{C}[x_1 \cdots, x_n]$ defined by $Q = \prod_{i>j}(x_i^e - x_j^e)$. Then $\deg Q = N$, and up to scalar, Q coincides with the product of the eigenvectors attached to all the reflections in W. It is easy to see that Q generates a one-dimensional representation of W affording ε_W. We define an operator $J : S(V) \to S(V)$ by

$$J = \sum_{w \in W} \varepsilon_W(w) w.$$

Then J is a projection on the ε_W-isotypic subspace of $S(V)$. We have the following remarkable result, although it is not used in the later discussion. Note that it is an analogue of [H, IV, Prop. 1.6].

Proposition 3.18. *Assume that $e \geq n$. Then there exists a non-zero constant d such that $\Delta_{w_1} = dQ^{-1}J$.*

Proof. It is known that S_W is a regular W-module, and S_W^N affords the sign representation of W. Hence we have

$$S^N(V) = (I_W)^N + \mathbb{C}Q,$$

where $(I_W)^N = I_W \cap S^N(V)$. Now $\widetilde{P}_{\lambda_0}Q_0 \in S^N(V)$, and (3.15.1) implies, in view of Lemma 3.16, that $\widetilde{P}_{\lambda_0}Q_0 \notin I_W$. Hence there exists a non-zero constant $c' \in \mathbb{C}$ such that $Q \equiv c'\widetilde{P}_{\lambda_0}Q_0 \pmod{I_W}$. In particular, we have $\Delta_{w_1}(Q) = c$ with $c = c'c_{\lambda_0}$, by Theorem 3.14. Since Δ_{w_1} and $Q^{-1}J$ are $S(V)^W$-endomorphisms of $S(V)$, both of them are determined by the restriction to $S^N(V)$. Hence, by comparing the value at Q, we see that $\Delta_{w_1} = dQ^{-1}J$ with $d = c/|W|$. This proves the proposition. □

3.19. We now return to the condition (3.12.2). We deduce several properties of the operators Δ_w by assuming this condition. Note that for any $\lambda, \mu \in \mathcal{M}_c$, the polynomial $\Delta_\lambda\Delta_{w_0}(\widetilde{P}_\mu Q_0)$ is a constant.
We denote by A_c the matrix $(\Delta_\lambda\Delta_{w_0}(\widetilde{P}_\mu Q_0))_{\lambda,\mu\in\mathcal{M}_c}$, under a suitable order, for a given integer $c \geq 0$. Then since $\Delta_\lambda\Delta_{w_0}(\widetilde{P}_\mu Q_0) = \Delta_\lambda(\widetilde{P}_\mu)$ by a similar argument as in (3.15.1), we see that

(3.19.1) Assume that (3.12.2) holds for W. Then the matrix A_c is non-singular.

We have the following lemma.

Lemma 3.20. *Assume that (3.12.2) holds for W. Then the operators $\{\Delta_\lambda\Delta_w |\ \lambda \in \mathcal{M},\ w \in S_n\}$ are linearly independent over $S(V)$.*

Proof. We consider the dependence relation

$$\sum_{\lambda,w} a(\lambda, w)\Delta_\lambda\Delta_w = 0 \tag{3.20.1}$$

on $S(V)$, where $a(\lambda, w) \in S(V)$. By induction on the length $\ell(w)$ of $w \in S_n$, we may assume that $a(\lambda, w') = 0$ for any $w' \in S_n$ such that $\ell(w') < \ell(w)$ and for $\lambda \in \mathcal{M}$. Multiplying $\Delta_{w^{-1}w_0}$ to the equation (3.20.1) from the right, and by making use of Proposition 2.7 together with induction hypothesis, we obtain

$$\sum_{\lambda\in\mathcal{M}} a(\lambda, w)\Delta_\lambda\Delta_{w_0} = 0. \tag{3.20.2}$$

We show that $a(\lambda, w) = 0$ by induction on the length of $\mathcal{M}$. Assume that $a(\mu', w) = 0$ for any $\mu' \in \mathcal{M}$ such that $\ell(\mu') < c$. We evaluate the equation (3.20.2) at $\widetilde{P}_\mu Q_0$ for $\mu \in \mathcal{M}_c$. Note that $\Delta_\lambda\Delta_{w_0}(\widetilde{P}_\mu Q_0) = 0$ if $\ell(\lambda) > c$.

Hence the non-zero contribution only comes from the terms corresponding to $\lambda \in \mathcal{M}_c$. We consider such equations for all $\mu \in \mathcal{M}_c$. Then it is regarded as a linear equation with variables $a(\lambda, w)$ $(\lambda \in \mathcal{M}_c)$, and with coefficient matrix A_c. Since the matrix A_c is non-singular by (3.19.1), we see that $a(\lambda, w) = 0$ for any $\lambda \in \mathcal{M}_c$. This proves the lemma. □

We can now prove the following proposition, which is analogous to proposition 2.14 in [RS].

Proposition 3.21. *Assume that (3.12.2) holds. Then the algebra $\mathcal{D}_W$ is a free $S(V)$-module with basis $\{\Delta_w \mid w \in W\}$.*

Proof. Let K be the quotient field of $S(V)$. The operator Δ_α on $S(V)$ can be extended to an operator on K. We consider the subalgebra $\mathcal{D}_W^K$ of $\operatorname{End}_{\mathbb{C}} K$ defined by $\mathcal{D}_W^K = K \otimes_{S(V)} \mathcal{D}_W$. Since $\dim_K \mathcal{D}_W^K \le |W|$, Lemma 3.20 implies that

(3.21.1) The set $\{\Delta_w \mid w \in W\}$ gives a basis of $\mathcal{D}_W^K$ as a K-vector space.

By a similar argument as in the proof of Lemma 2.14 in [RS], the proof of the proposition is reduced to showing the following lemma.

Lemma 3.22. *Let Δ be a d-product of Δ_s $(s \in S)$. Then Δ can be written as*

$$\Delta = \sum_{w \in W} a_w \Delta_w,$$

where $a(w)$ are elements in $S(V)$ satisfying the following conditions.

$$\begin{cases} a_w = 0 & \text{if } \ell(w) < d, \\ a_w \in S^{\ell(w)-d}(V) & \text{if } \ell(w) \ge d. \end{cases} \tag{3.22.1}$$

We prove Lemma 3.22. Here we recall that any $\Delta_{w'}$ $(w' \in W)$ can be written as $\Delta_{w'} = \Delta_\lambda \Delta_w$ with $\lambda \in \mathcal{M}$, $w \in S_n$. Hence by (3.21.1) Δ can be expressed as

$$\Delta = \sum_{\substack{\lambda \in \mathcal{M} \\ w \in S_n}} a(\lambda, w) \Delta_\lambda \Delta_w, \tag{3.22.2}$$

with $a(\lambda, w) \in K$. We write $a(\lambda, w) = a_{w'}$ if $w' \in W$ corresponds to (λ, w). We shall prove that $a(\lambda, w)$ satisfies the condition (3.22.1) by induction on the length $\ell(\lambda)$ of $\mathcal{M}$, and on the length $\ell(w)$ of S_n. We fix $w \in S_n$ and assume that (3.22.1) is verified for any $a(\lambda', w')$ such that $\lambda' \in \mathcal{M}$ and that $w' \in S_n$ with $\ell(w') < \ell(w)$. Also we assume that it is verified for any $a(\mu', w)$ such that $\ell(\mu') < c$ for an integer $c \geq 0$. We show that $a(\lambda, w)$ satisfies (3.22.1) for any $\lambda \in \mathcal{M}_c$. By multiplying $\Delta_{w^{-1}w_0}$ on both sides of (3.22.2) from the right, we have

$$\Delta\Delta_{w^{-1}w_0} = \sum_{\lambda \in \mathcal{M}} a(\lambda, w)\Delta_\lambda \Delta_{w_0} + \sum_{\lambda', w'} a(\lambda', w')\Delta_{\lambda'}\Delta_{w''}, \tag{3.22.3}$$

where in the second sum, λ' runs over all the elements in $\mathcal{M}$, and w' in S_n such that $\ell(w') < \ell(w)$. Here $w'' \in S_n$ is given by $w'' = w'w^{-1}w_0$ with $\ell(w'') = \ell(w') - \ell(w) + \ell(w_0)$. We evaluate the equation (3.22.3) at $\widetilde{P}_\mu Q_0$, with $\mu \in \mathcal{M}_c$, which is a polynomial of degree $c + \ell(w_0)$. Then the non-zero contribution in the first sum comes from the terms corresponding to $\lambda \in \mathcal{M}_1$, where

$$\mathcal{M}_1 = \{\lambda \in \mathcal{M} |\ \ell(\lambda) \leq c\}.$$

First assume that $c + \ell(w) < d$. Then for any $\lambda \in \mathcal{M}_1$, we have $\ell(\lambda) + \ell(w) < d$. Hence by induction hypothesis, we have $a(\lambda, w) = 0$ for $\lambda \in \mathcal{M}_1$ such that $\ell(\lambda) < c$. On the other hand, again by induction hypothesis, $a(\lambda', w')\Delta_{\lambda'}\Delta_{w''}(\tilde{P}_\mu Q_0)$ is a homogeneous polynomial of degree $c+\ell(w)-d < 0$. This means that there are no contributions from the terms in the second sum, and we have

$$\Delta\Delta_{w^{-1}w_0}(\widetilde{P}_\mu Q_0) = \sum_{\lambda \in \mathcal{M}_c} a(\lambda, w)\Delta_\lambda \Delta_{w_0}(\tilde{P}_\mu Q_0).$$

Since $d + \ell(w^{-1}w_0) > \ell(\mu) + \ell(w_0)$, we have $\Delta\Delta_{w^{-1}w_0}(\widetilde{P}_\mu Q_0) = 0$. This implies that $a(\lambda, w) = 0$ for any $\lambda \in \mathcal{M}_c$, since the matrix A_c is non-singular by (3.19.1). Next assume that $c + \ell(w) \geq d$. Take $\lambda \in \mathcal{M}$ such that $\ell(\lambda) < c$. Then by induction hypothesis, $a(\lambda, w)$ is a homogeneous polynomial of degree $\ell(\lambda)+\ell(w)-d$ for such λ, if it is positive, and $a(\lambda, w) = 0$ if $\ell(\lambda)+\ell(w)-d < 0$. Hence $a(\lambda, w)\Delta_\lambda\Delta_{w_0}(\widetilde{P}_\mu Q_0)$ is a homogeneous polynomial of degree $c+\ell(w)-d$, if it is non-zero. On the other hand, by a similar argument as before we see that the term in the second sum $a(\lambda', w')\Delta_{\lambda'}\Delta_{w''}(\tilde{P}_\mu Q_0)$ is also a homogeneous polynomial of degree $c+\ell(w)-d$, if it is non-zero. Moreover, $\Delta\Delta_{w^{-1}w_0}(\widetilde{P}_\mu Q_0)$ is a homogeneous polynomial of the same degree. Since the matrix A_c is a non-singular $\mathbb{C}$-matrix, we see that $a(\lambda, w)$ is a homogeneous polynomial of degree $c + \ell(w) - d$ for any $\lambda \in \mathcal{M}_c$. This shows that $a(\lambda, w)$ satisfies the condition in (3.22.1). The lemma is now proved and the proposition follows. □

The following lemma can be proved in a similar way as Lemma 2.16 in [RS], in view of [RS, Remark 2.10].

Lemma 3.23. *Let P be a homogeneous polynomial of degree N. Let I be a graded ideal of $S(V)$ containing I_W, but not containing P. Then $I = I_W$.*

3.24 Let $S(V)^*$ be the graded vector space defined by $S(V)^* = \oplus_{i\geq 0} S^i(V)^*$, where $S^i(V)^*$ denotes the dual space of $S^i(V)$ over $\mathbb{C}$. We have a natural pairing $<,>: S(V) \times S(V)^* \to \mathbb{C}$, $< u, f >= f(u)$. Let $\varepsilon : S(V) \to \mathbb{C}$ denote the evaluation at 0. Then for each $\Delta \in \mathcal{D}_W$ we can regard $\varepsilon\Delta$ as an element in $S(V)^*$. Let $\bar{\mathcal{D}}_W$ be the subspace of $S(V)^*$ generated by $\varepsilon\Delta$ with $\Delta \in \mathcal{D}_W$. Let H_W be the dual space of $\bar{\mathcal{D}}_W$. Then we have a natural map $c : S(V) \to H_W$, which sends $u \in S(V)$ to the restriction to $\bar{\mathcal{D}}_W$ of the map $< u, \cdot >: S(V) \to \mathbb{C}$. We can now state the main theorem, which is an analogue of [RS. Th. 2.18].

Theorem 3.25. *Assume that the conjectures (3.12.1) and (3.12.2) hold for W. Then there exists a unique graded $\mathbb{C}$-algebra structure on H_W such that c induces an isomorphism $S_W \cong H_W$. The set $\{\varepsilon\Delta_w | w \in W\}$ gives a basis of the $\mathbb{C}$-vector space $\bar{\mathcal{D}}_W$. In particular, if we denote by $\{X_w | w \in W\}$ the dual basis of $\{\varepsilon\Delta_w | w \in W\}$, the map c can be described, for $u \in S(V)$, as*

$$c(u) = \sum_{w\in W} \varepsilon\Delta_w(u) X_w.$$

Proof. It follows from proposition 3.21 that $\{\varepsilon\Delta_w | w \in W\}$ gives rise to a basis of $\bar{\mathcal{D}}_W$. Since $\dim S_W = |W|$, in order to prove the theorem it is enough to prove that $\operatorname{Ker} c = I_W$. Since $\mathcal{D}_W$ has a structure of a right $S(V)$-module, we see that $\operatorname{Ker} c$ is a graded ideal of $S(V)$. It also follows from Lemma 3.16 that $I_W \subset \operatorname{Ker} c$. Now (3.12.1) asserts that $\Delta_{\lambda_0}\Delta_{w_0}(\widetilde{P}_{\lambda_0} Q_0) \neq 0$ (see (3.15.1)). Hence $\widetilde{P}_{\lambda_0} Q_0$ is a polynomial with $\deg \widetilde{P}_{\lambda_0} Q_0 = N$, which is not contained in I. Then one can apply Lemma 3.23 with $P = \widetilde{P}_{\lambda_0} Q_0$ and we conclude that $I = I_W$. This proves the theorem. □

References

[BGG] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand; Schubert cells and cohomology of the space G/P, Russian Math. Surveys **28** (1973), 1–26. Also in "Representation theory", London Math. Soc. Lecture Note Series **69**, pp. 115–140, Cambridge Univ. Press, Cambridge 1982.

[BM1] K. Bremke and G. Malle, *Reduced words and a length function for $G(e, 1, n)$*, Indag. Math. **8** (1997), 453-469.

[BM2] K. Bremke and G. Malle, *Root systems and length functions*, Geometriae Dedicata **72** (1998), 83-97.

[D] M. Demazure, *Invariants symétriques des groupes de Weyl et torsion*, Inv. Math. **21**(1973), 287-301.

[H] H.L. Hiller, *Geometry of Coxeter groups*, Research Notes in Mathematics, No.54, Pitman, Boston 1982.

[RS] K. Rampetas and T. Shoji, *Length functions and Demazure operators for* $G(e,1,n)$, *I and II*, to appear in Indag. Math.

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